Linear Diophantine equations on certain sparse sets
(疎な集合上の線形ディオファントス方程式について)

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※ p.6 Corollary が間違っていました.
There is a mistake in Corollary on p.6.

§1 Introduction

Introduction

We begin with the following equation:

$$x + y = z. \tag{1}$$

There are infinitely many tuples $(x, y, z) \in \mathbb{N}^3$ satisfying (1). For example,

$$(x, y, z) = (1, 1, 2),\ (1, 3, 4),\ (1, 4, 5),\ (2, 5, 7),\ \cdots .$$

Let $\mathrm{P}(a) = \{n^a \colon n \in \mathbb{N}\}$ for every $a \in \mathbb{N}$.

Question 1

Given $a \in \mathbb{N}$, does there exist $(x, y, z) \in \mathrm{P}(a)^3$ satisfying (1)?

In the case $a = 2$, $(x, y, z) \in \mathrm{P}(2)^3$ satisfying (1) is called a Pythagorean triple. **There are infinitely many such triples**. For example,

$$(x, y, z) = (9, 16, 25),\ (25, 144, 169),\ (64, 225, 289),\ \cdots .$$

Introduction

Theorem (Fermat's last theorem (finally solved by Wiles in 1995))

If $a \in \mathbb{N}$ is greater than or equal to 3, then there does not exist $(x, y, z) \in \mathrm{P}(a)^3$ satisfying $x + y = z$.

Main problem

$$\text{Problems on } \mathbb{N} \longrightarrow \text{Problems on } \mathbb{R}$$

Let $\lfloor x \rfloor$ be the integer part of $x \in \mathbb{R}$.

Definition

For every non-integral $\alpha > 1$, $(\lfloor n^\alpha \rfloor)_{n \in \mathbb{N}}$ is called the Piatetski-Shapiro sequence with exponent α. Further, we define

$$\mathrm{PS}(\alpha) = \{\lfloor n^\alpha \rfloor : n \in \mathbb{N}\}.$$

Piatetski-Shapiro sequences

Let $\mathrm{PS}(\alpha) = \{\lfloor n^\alpha \rfloor : n \in \mathbb{N}\}$.

Question 2

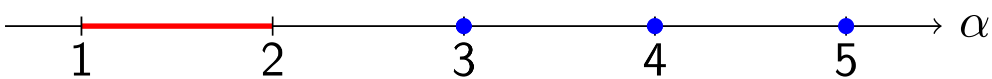
When does $x + y = z$ have a solution $(x, y, z) \in \mathrm{PS}(\alpha)^3$?

- $\mathrm{PS}(1.2) = \{1, 2, 3, 5, 6, 8, 10, 12, 13, 15, \ldots\}$;
- $\mathrm{PS}(2.2) = \{1, 4, 11, 21, 34, 51, 72, 97, 125, 158, \ldots\}$;
- $\mathrm{PS}(3.2) = \{1, 9, 33, 84, 172, 309, 506, 776, 1131, 1584, \ldots\}$.

  There is no solution $(x, y, z) \in (\mathrm{PS}(3.2) \cap [1, 6.30 \times 10^{12}])^3$.*

Theorem (It follows from [Frantikinakis & Wierdl, 2009, Adv. Math.])

$\forall \alpha \in (1, 2)$, $\exists^\infty (x, y, z) \in \mathrm{PS}(\alpha)^3$ s.t. $x + y = z$.



*This was computed by Matsusaka. It can be seen in https:// www.sci.kyushu-u.ac.jp/koho/qrinews/qrinews_220609.html

Piatetski-Shapiro sequences

Main theorem I (Matsusaka & S., 2021, Acta Arith.)

For all $2 < s < t$, $\{\alpha \in [s,t] \colon \exists^\infty (x,y,z) \in \mathrm{PS}(\alpha)^3 \text{ s.t. } x+y=z\}$ is uncountable.

Corollary

~~There are at most countably many $\alpha \in [1,\infty)$ such that $x+y=z$ does not have any solutions $(x,y,z) \in \mathrm{PS}(\alpha)^3$.~~

Proof) If $\lfloor p^\alpha \rfloor + \lfloor q^\alpha \rfloor = \lfloor r^\alpha \rfloor$, then there exists $\ell = \ell(\alpha,p,q,r) > 0$ such that for all $\tau \in (\alpha, \alpha+\ell)$ we have $\lfloor p^\tau \rfloor + \lfloor q^\tau \rfloor = \lfloor r^\tau \rfloor$.

$$\{\alpha \in [1,\infty) \colon \exists (x,y,z) \in \mathrm{PS}(\alpha)^3 \text{ s.t. } x+y=z\}$$
$$\supseteq \bigcup_{\substack{\alpha \in [1,\infty) \text{ satisfying} \\ \exists (p,q,r) \lfloor p^\alpha \rfloor + \lfloor q^\alpha \rfloor = \lfloor r^\alpha \rfloor}} (\alpha, \alpha+\ell)$$

This is incorrect. Indeed, let F be the Cantor middle third set. Then F^c is open and dense in $\mathbb{R}$. However, $(F^c)^c$ is uncountable.

which is open and dense in $[1,\infty)$. ~~The complement is at most countable.~~

§2 Motivation and Background

Motivation

A sequence of real numbers $(a_i)_{i=0}^{k-1}$ is called an *arithmetic progression of length k* (*k-AP* for short) if there exist $a \in \mathbb{R}$ and $d > 0$ such that

$$a_i = a + id \quad (i = 0, 1, \ldots, k-1).$$

Problem

Fix any $k \geq 3$. If $A \subseteq \mathbb{N}$ is given, then does A contain a k-AP, or not?

For example, choose $A = \mathrm{P}(2) = \{n^2 \colon n \in \mathbb{N}\}$.

$$A = \{1, \quad 4, \quad 9, \quad 16, \quad 25, \quad 36, \quad 49, \quad 64, \ \ldots\}$$

- 3-APs: $(1, 25, 49) \longrightarrow (n^2, (5n)^2, (7n)^2)$ for every $n = 1, 2, \ldots$
- 4-APs: Euler showed that A does not contain any 4-APs in 1780.

Arithmetic progressions

Ramsey theory

Large size $\Rightarrow$ Existence of a given structure

Define $[N] = \{1, 2, \ldots, N\}$.

Szemerédi, 1975, Acta Arith.

If $A \subseteq \mathbb{N}$ satisfies

$$\limsup_{N\to\infty} \#(A \cap [N])/N > 0,$$

then A contains arbitrarily long APs.

Question 4

Fix any $k \geq 3$. Given a set $A \subseteq \mathbb{N}$ satisfying $\limsup_{n\to\infty} \#(A \cap [N])/N = 0$, does A contain a k-AP, or not?

Arithmetic progressions of primes

Let $\mathcal{P}$ be the set of all prime numbers. By the prime number theorem, there exists a absolute constant $C > 0$ such that

$$\#(\mathcal{P} \cap [N])/N \leq C/\log N.$$

Green & Tao, 2008, Ann. of Math. (2)

The set of all prime numbers contains arbitrarily long APs.

More precisely, Green and Tao showed the following much stronger result.

Green & Tao, 2008, Ann. of Math. (2)

Let $A \subseteq \mathcal{P}$. If A satisfies

$$\limsup_{N\to\infty} \#(A \cap [N])/\#(\mathcal{P} \cap [N]) > 0,$$

then A contains arbitrarily long APs.

Arithmetic progressions of Piatetski-Shapiro sequences

Let $\alpha > 1$ be a non-integral real number. For all $N \in \mathbb{N}$, we have

$$\#(\mathrm{PS}(\alpha) \cap [N]) = \#\{n \in \mathbb{N} \colon \lfloor n^\alpha \rfloor \le N\} \approx N^{1/\alpha}.$$

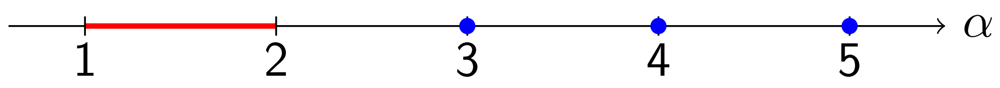
We can not apply Szemerédi's theorem to Piatetski-Shapiro sequences.

Theorem (It follows from [Frantikinakis & Wierdl, 2009, Adv. Math.])

For every $1 < \alpha < 2$, $\mathrm{PS}(\alpha)$ contains arbitrarily long APs.

Darmon & Merel, 1997 (partially by Euler, Legendre, and Dénes)

For all integers $a \ge 3$, $\mathrm{P}(a) = \{n^a \colon n \in \mathbb{N}\}$ does not contain any 3-APs.



Szemerédi's theorem on Piatetski-Shapiro sequences

S. & Yoshida, 2019, J. Integer Seq.

Let $1 < \alpha < 2$. Let A be a subset of $\mathbb{N}$ satisfying $\limsup_{n\to\infty} \dfrac{\#(A \cap [N])}{N} > 0$. Then $\{\lfloor n^\alpha \rfloor : n \in A\}$ contains arbitrarily long APs.

This theorem can be considered as Szemerédi's theorem on Piatetski-Shapiro sequences. Indeed, this theorem implies

Corollary

Let $1 < \alpha < 2$, and let $A \subseteq \mathrm{PS}(\alpha)$. If A satisfies

$$\limsup_{N\to\infty} \#(A \cap [N])/\#(\mathrm{PS}(\alpha) \cap [N]) > 0,$$

then A contains arbitrarily long APs.

$$\lfloor n^\alpha \rfloor = n^\alpha + \underline{\underline{O(1)}}$$

$$\shortparallel$$

$$\alpha(\alpha-1)\,n^{\underline{\alpha-2}}$$

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§3 Main theorem II

Main theorem II

Matsusaka & S., 2021, Acta Arith.

For all $2 < s < t$, $\{\alpha \in [s,t] \colon \mathrm{PS}(\alpha)$ contains infinitely many 3-APs$\}$ is uncountable.

Main theorem II (Matsusaka & S., 2021, Acta Arith.)

Let $a, b, c \in \mathbb{N}$. For all $2 < s < t$, we have

$$\dim_{\mathrm{H}}\{\alpha \in [s,t] \colon ax + by = cz \text{ has infinitely many solutions } (x,y,z) \in \mathrm{PS}(\alpha)^3 \text{ with } \#\{x,y,z\} = 3\} \geq 1/s^3$$

Note that $\dim_{\mathrm{H}} F > 0$ implies F is uncountable.

Main theorem I (Recall)

For all $2 < s < t$, $\{\alpha \in [s,t] \colon \exists^{\infty}(x,y,z) \in \mathrm{PS}(\alpha)^3 \text{ s.t. } x + y = z\}$ is uncountable.

Sketch of the proof of Main theorem II

Fix any $a, b, c \in \mathbb{N}$, and real numbers $2 < s < t$. Fix $x \in \mathbb{N}$. Let $J(x) \subset \mathbb{N}$ be a certain finite interval. Fix $z \in J(x)$. Take a certain $y = y(x, z)$.

Step 1. The intermediate value theorem $\Rightarrow \exists \alpha = \alpha(x, y, z) > 0$ such that

$$ax^\alpha + by^\alpha = cz^\alpha.$$

Step 2. For $n \in \mathbb{N}$, we observe that

$$\begin{aligned}
&|a\lfloor (nx)^\alpha \rfloor + b\lfloor (ny)^\alpha \rfloor - c\lfloor (nz)^\alpha \rfloor| \\
&\le |a(nx)^\alpha + b(ny)^\alpha - c(nz)^\alpha| + |a\{(nx)^\alpha\} + b\{(ny)^\alpha\} - c\{(nz)^\alpha\}| \\
&= |a\{(nx)^\alpha\} + b\{(ny)^\alpha\} - c\{(nz)^\alpha\}|.
\end{aligned}$$

By the theory of uniform distribution, we find $n = n(x, y, z) \in \mathbb{N}$ such that

$$|a\{(nx)^\alpha\} + b\{(ny)^\alpha\} - c\{(nz)^\alpha\}| < 1.$$

Therefore for such n, we have

$$c\lfloor (nz)^\alpha \rfloor = a\lfloor (nx)^\alpha \rfloor + b\lfloor (ny)^\alpha \rfloor.$$

Sketch of the proof of Main theorem II

By the above discussion, for all $x \in \mathbb{N}$ and $z \in J(x)$, there exist $\alpha = \alpha(x, y, z)$ and $n = n(x, y, z)$

$$c\lfloor (nz)^\alpha \rfloor = a\lfloor (nx)^\alpha \rfloor + b\lfloor (ny)^\alpha \rfloor.$$

Step 3. Find $\ell = \ell(x, y, z) > 0$ such that for all $\tau \in (\alpha, \alpha + \ell)$

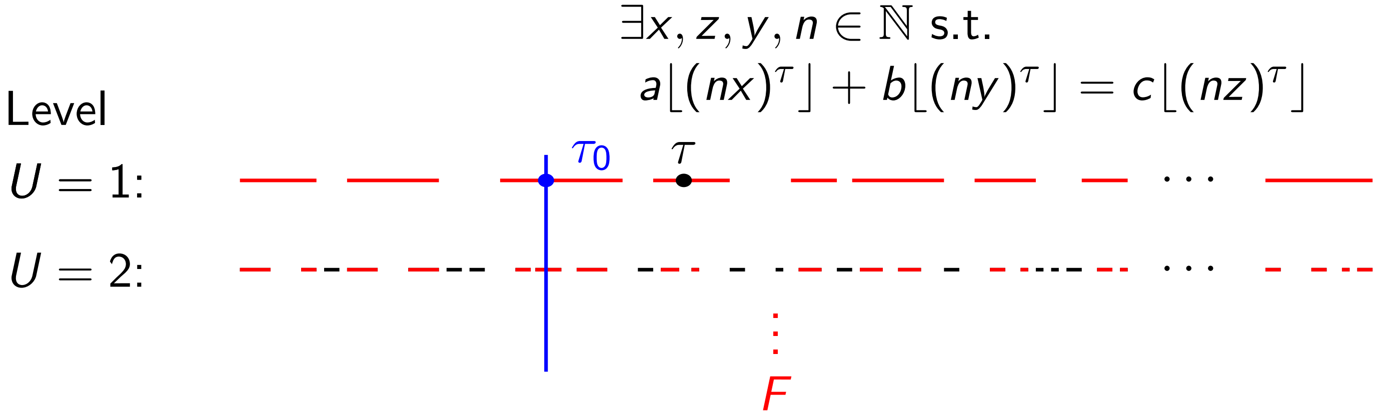
$$\lfloor (nx)^\alpha \rfloor = \lfloor (nx)^\tau \rfloor, \ \lfloor (ny)^\alpha \rfloor = \lfloor (ny)^\tau \rfloor, \ \lfloor (nz)^\alpha \rfloor = \lfloor (nz)^\tau \rfloor.$$

Step 4. Define

$$F := \bigcap_{U=1}^{\infty} \bigcup_{U < x \leq 2U} \bigcup_{z \in J(x)} (\alpha(x, y, z), \alpha(x, y, z) + \ell(x, y, z)).$$

$$F := \bigcap_{U=1}^{\infty} \bigcup_{U<x\leq 2U} \bigcup_{z\in J(x)} (\alpha(x,y,z), \alpha(x,y,z)+\ell(x,y,z)).$$

Step 5.



Take any $\tau_0 \in F$. Then for every $U \in \mathbb{N}$ there exist $x_U, z_U, y_U, n_U \in \mathbb{N}$ such that $a\lfloor (n_U x_U)^{\tau_0} \rfloor + b\lfloor (n_U y_U)^{\tau_0} \rfloor = c\lfloor (n_U z_U)^{\tau_0} \rfloor$. Therefore

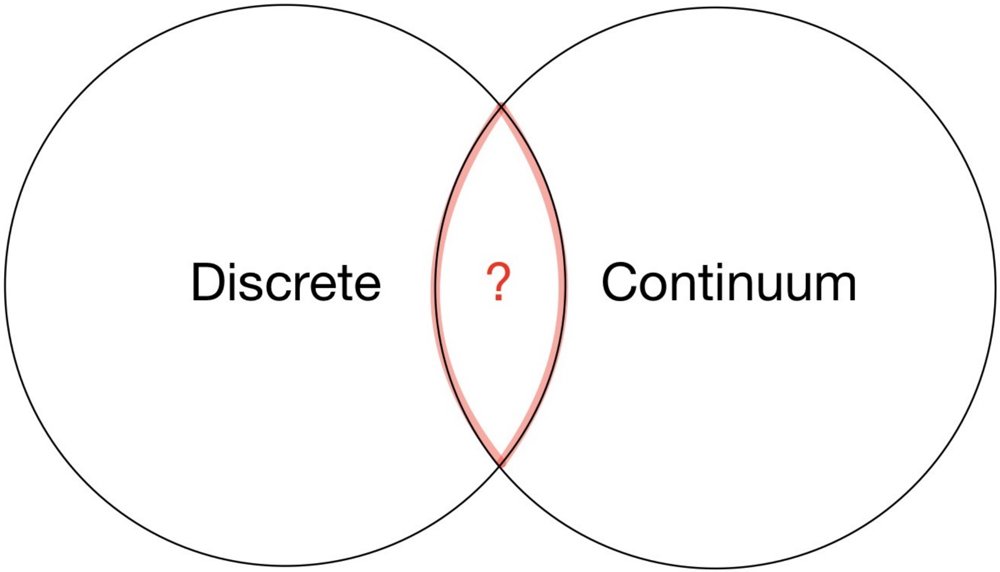
$$F \subseteq \{\alpha \in [s,t] \colon ax + by = cz \text{ has infinitely many solutions } (x,y,z) \in \mathrm{PS}(\alpha)^3 \text{ with } \#\{x,y,z\} = 3\}.$$

We calculate $\dim_{\mathrm{H}} F$ by a classical method for a general Cantor set.

Intermediates between discrete and continuum

Main problem

Problems on $\mathbb{N} \longrightarrow$ Problems on $\mathbb{R}$



§5 Further researches

Glasscock's results

Glasscock, 2017 & 2020, Acta Arith.

Suppose $a, b \in \mathbb{R}$, $a \notin \{0, 1\}$ satisfy that the equation (E) $y = ax + b$ has infinitely many solutions $(x, y) \in \mathbb{N}^2$. For Lebesgue almost all $\alpha > 1$,

- if $\alpha < 2$, (E) has infinitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$;
- if $\alpha > 2$, (E) has at most finitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$.

Glasscock, 2017, Acta Arith.

For Lebesgue almost all $1 < \alpha < 2$, there exist infinitely many $(k, m, \ell) \in \mathbb{N}^3$ such that all of

$$k, \quad m, \quad \ell, \quad k+m, \quad m+\ell, \quad \ell+k, \quad k+m+\ell$$

are in $\mathrm{PS}(\alpha)$.

Perfect Euler brick

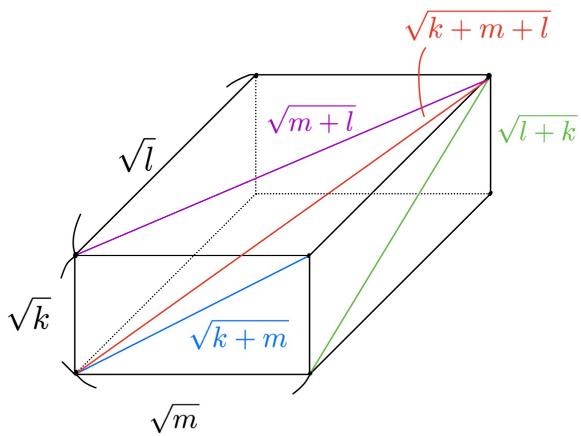
Problem involving the existence of a perfect Euler brick

Does there exists $(k, \ell, m) \in \mathbb{N}^3$ such that all of

$$k, \quad m, \quad \ell, \quad k+m, \quad m+\ell, \quad \ell+k, \quad k+m+\ell \tag{2}$$

are in $\mathrm{P}(2)(=\mathrm{PS}(2))$?

If there was such a tuple $(k, \ell, m) \in \mathbb{N}^3$, we would prove the existence of a perfect Euler brick.



Improvement of Glasscock's result

Glasscock, 2017 & 2020, Acta Arith.

Suppose $a, b \in \mathbb{R}$ with $a \notin \{0, 1\}$ satisfy that the equation (E) $y = ax + b$ has infinitely many solutions $(x, y) \in \mathbb{N}^2$. For Lebesgue almost all $\alpha > 1$,

- if $\alpha < 2$, (E) has infinitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$,
- if $\alpha > 2$, (E) has at most finitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$.

S., 2022, Acta Arith.

Suppose $a, b \in \mathbb{R}$ with $a \neq 1$ and $0 \leq b < a$ satisfy that (E) has infinitely many solution $(x, y) \in \mathbb{N}^2$. Then

- for all $1 < \alpha < 2$, (E) has infinitely many solutions $(x, y) \in \mathrm{PS}(\alpha)^2$;
- for all real numbers $2 < s < t$,

$$\dim_{\mathrm{H}}\{\alpha \in (s, t) \colon \text{(E) has infinitely many solutions } (x, y) \in \mathrm{PS}(\alpha)^2\} = 2/s.$$

Improvement of Glasscock's result

S., 2022, Acta Arith.

For all $1 < \alpha < 2$, there exist infinitely many $(k, m, \ell) \in \mathbb{N}^3$ such that all of

$$k, \quad m, \quad \ell, \quad k+m, \quad m+\ell, \quad \ell+k, \quad k+m+\ell$$

are in $\mathrm{PS}(\alpha)$.

Let $S(\alpha) = \{\lfloor \alpha n^2 \rfloor : n \in \mathbb{N} \text{ and } n \geq \alpha^{-1/2}\}$ for all $\alpha \in (0, 1]$.

Kanado & S., 2022+, arXiv:2205.12226

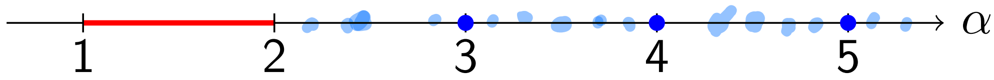
For Lebesgue almost all $0 < \alpha < 1$, there exist infinitely many $(k, m, \ell) \in \mathbb{N}^3$ such that all of

$$k, \quad m, \quad \ell, \quad k+m, \quad m+\ell, \quad \ell+k, \quad k+m+\ell$$

are in $S(\alpha)$.

Summary

- We discussed the set of α such that there are infinitely many $(x, y, z) \in \mathrm{PS}(\alpha)^3$ such that $x + y = z$ as follows:



- This research is motivated by problems of APs.

  "Large size $\Rightarrow$ Existence of the given structures"

- For all $2 < s < t$, we have

$$\dim_{\mathrm{H}}\{\alpha \in [s, t] \colon ax + by = cz \text{ has infinitely many solutions } (x, y, z) \in \mathrm{PS}(\alpha)^3 \text{ with } \#\{x, y, z\} = 3\} \geq 1/s^3.$$

- We discussed similar problems with $y = ax + b$ and perfect Euler bricks.

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