Limit theorems for the value-distributions of zeta-functions

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Zeta and L-functions

There are various zeta and L-functions arising from various mathematical objects.

- Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

- L-functions of cusp forms, e.g.

$$L(s,\Delta) = \sum_{n=1}^{\infty} \tau(n) n^{-s} \qquad \tau(1) = 1,\ \tau(2) = -24,\ \tau(3) = 252, \ldots$$

Here, $\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$ with $q = e^{2\pi i z}$ for $\mathrm{Im}(z) > 0$.

- Dirichlet L-function $L(s,\chi)$, Dedekind zeta-function $\zeta_K(s)$, Hurwitz zeta-function $\zeta(s,\alpha)$, etc.

Limit theorems

Central limit theorem (CLT)

Let $X_1, X_2, \ldots$ be i.i.d. random variables such that $\mathbb{E}[X_1] = 0$ and $\mathrm{Var}[X_1] = 1$. Then we have

$$\lim_{n\to\infty} \mathbb{P}\left(a \leq \frac{X_1 + \cdots + X_n}{\sqrt{n}} \leq b\right) = \int_a^b \exp\left(-\frac{x^2}{2}\right) \frac{dx}{\sqrt{2\pi}}$$

for any real numbers $a < b$.

Today's talk:

We consider that values of zeta and L-functions behave as if they were random variables.

values of zeta and L-functions	$\longleftrightarrow$	random variables
limit theorems	$\longleftrightarrow$	CLT
M-functions	$\longleftrightarrow$	$\exp\left(-\frac{x^2}{2}\right)$

Contents

Riemann zeta-function

Let $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$. We define

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

- One can extend $\zeta(s)$ as a meromorphic function on $\mathbb{C}$.
- For $\sigma > 1$, we have

$$\zeta(s) = \prod_{p:\ \text{prime}} (1 + p^{-s} + p^{-2s} + \cdots) = \prod_{p:\ \text{prime}} (1 - p^{-s})^{-1}.$$

- Properties of $\zeta(s)$ are related to the distribution of prime numbers.

Limit theorem for $\zeta(\sigma+it)$ in t-aspect

Bohr–Jessen's limit theorem (one-dimensional)

Let $\sigma > 1/2$. Then there exists a continuous function $M_\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{t \in [0,T] \,\middle|\, a \leq \log|\zeta(\sigma+it)| \leq b\right\} = \int_a^b M_\sigma(x) \frac{dx}{\sqrt{2\pi}}$$

for any real numbers $a < b$.

- Bohr–Jessen (1930,1932) originally stated a two-dimensional limit theorem by considering

$$\log\zeta(\sigma+it) \in [a,b] \times i[c,d]$$

with a certain branch of logarithm of $\zeta(s)$.

- RHS of the limit theorem is represented as

$$\int_a^b M_\sigma(x)\,\frac{dx}{\sqrt{2\pi}} = \mathbb{P}\big(a \le \log|\zeta(\sigma, X)| \le b\big)$$

by using the following random Euler product $\zeta(\sigma, X)$.

Random Euler product (1)

For $\sigma > 1/2$, we define

$$\zeta(\sigma, X) = \prod_p (1 - p^{-\sigma} X_p)^{-1},$$

where $X = \{X_p\}_p$ is a sequence of independent random variables distributed on the unit circle according to the normalized Haar measure, i.e.

$$\mathbb{P}\left(\alpha \le \arg X_p \le \beta\right) = \frac{\beta - \alpha}{2\pi}, \qquad 0 \le \alpha < \beta \le 2\pi.$$

- One can show that $M_\sigma(x) > 0$ for all $x \in \mathbb{R}$ if $1/2 < \sigma \leq 1$. Hence we obtain the following.

Corollary

Let $1/2 < \sigma \leq 1$. Then we have

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{t \in [0,T] \;\middle|\; \Big|\log|\zeta(\sigma+it)| - a\Big| < \epsilon\right\} = \int_{a-\epsilon}^{a+\epsilon} M_\sigma(x) \frac{dx}{\sqrt{2\pi}} > 0$$

for any $a \in \mathbb{R}$ and $\epsilon > 0$.
In other words, there are many real numbers t such that $\log|\zeta(\sigma+it)|$ approximates $a \in \mathbb{R}$.

Remark: Bohr–Courant (1914) proved that the set $\{\zeta(\sigma+it) \mid t \in \mathbb{R}\}$ is dense in $\mathbb{C}$ if $1/2 < \sigma \leq 1$.

L-functions of cusp forms

Let q be a large prime number.

$$S_2(q) = \text{ the space of holomorphic cusp forms for } \Gamma_0(q) \text{ of weight } 2 \text{ with trivial nebentypus}$$
$$B_2(q) = \text{ a basis of } S_2(q) \text{ consisting of primitive forms}$$

Let $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$. For $f \in S_2(q)$, we define

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s},$$

where $f(z) = \sum_{n=1}^{\infty} \lambda_f(n)\sqrt{n}\, e^{2\pi i n z}$ for $\mathrm{Im}(z) > 0$.

- One can extend $L(s, f)$ as a holomorphic function on $\mathbb{C}$.
- If $f \in B_2(q)$, then we have

$$L(s, f) = (1 - \lambda_f(q) q^{-s})^{-1} \prod_{p \neq q} (1 - \lambda_f(p) p^{-s} + p^{-2s})^{-1}.$$

Limit theorem for $L(\sigma, f)$ in level-aspect

Golubeva's limit theorem

Let $\sigma > 1/2$. Then there exists a continuous function $\mathscr{M}_\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

$$\lim_{q\to\infty} \frac{1}{\#B_2(q)} \#\{f \in B_2(q) \mid a \leq \log L(\sigma, f) \leq b\} = \int_a^b \mathscr{M}_\sigma(x) \frac{dx}{\sqrt{2\pi}}$$

for any real numbers $a < b$.

- Golubeva (2004) proved this result only for $\sigma = 1$, but the case $\sigma > 1$ follows immediately from the method of Golubeva. To show the case $1/2 < \sigma < 1$, we apply the zero-density estimate due to Kowalski–Michel (1999).
- Cogdell–Michel (2004) also proved a similar limit theorem weighted by $\omega_f := 1/4\pi\langle f, f\rangle$.

- RHS of the limit theorem is represented as

$$\int_a^b \mathscr{M}_\sigma(x)\,\frac{dx}{\sqrt{2\pi}} = \mathbb{P}\big(a \le \log L(\sigma, Y) \le b\big)$$

by using the following random Euler product $L(\sigma, Y)$.

Random Euler product (2)

For $\sigma > 1/2$, we define

$$L(\sigma, Y) = \prod_p (1 - 2\cos(Y_p)p^{-\sigma} + p^{-2\sigma})^{-1},$$

where $Y = \{Y_p\}_p$ is a sequence of independent random variables distributed on $[0, \pi]$ according to the p-adic Plancherel measure, i.e.

$$\mathbb{P}\left(\alpha \le Y_p \le \beta\right) = \int_\alpha^\beta \frac{1 + \frac{1}{p}}{1 - \frac{2\cos 2\theta}{p} + \frac{1}{p^2}}\,\frac{2}{\pi}\sin^2\theta\,d\theta, \qquad 0 \le \alpha < \beta \le \pi.$$

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Discrepancy estimate in CLT

Let $X_1, X_2, \dots$ be i.i.d. random variables such that $\mathbb{E}[X_1] = 0$ and $\mathrm{Var}[X_1] = 1$.

$$\text{CLT :} \qquad \mathbb{P}\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}} \in \mathcal{I} \right) \to \int_{\mathcal{I}} \exp\left(-\frac{x^2}{2} \right) \frac{dx}{\sqrt{2\pi}} \qquad \mathcal{I} = [a,b] \subset \mathbb{R}$$

Berry–Esseen's theorem

Suppose that $\beta = \mathbb{E}[|X_1|^3]$ is finite. Then there exists a constant $C > 0$ such that

$$\sup_{\mathcal{I} \subset \mathbb{R}} \left| \mathbb{P}\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}} \in \mathcal{I} \right) - \int_{\mathcal{I}} \exp\left(-\frac{x^2}{2} \right) \frac{dx}{\sqrt{2\pi}} \right| \leq C \frac{\beta}{\sqrt{n}}.$$

- $C \leq 7.59$ (Esseen, 1942), $C \geq 0.409$ (Esseen, 1956), $C \leq 0.474$ (Shevtsova, 2013)

Discrepancy estimate for $\zeta(\sigma + it)$

Let $\sigma > 1/2$ and $\mathcal{I} = [a, b] \subset \mathbb{R}$. Then we define

$$D_\sigma(T;\mathcal{I}) = \left| \frac{1}{T} \operatorname{meas}\left\{ t \in [0,T] \;\middle|\; \log|\zeta(\sigma+it)| \in \mathcal{I} \right\} - \int_{\mathcal{I}} M_\sigma(x)\,\frac{dx}{\sqrt{2\pi}} \right|.$$

Put $|\mathcal{I}| = b - a$. For $1/2 < \sigma \leq 1$ and $\epsilon > 0$, the following estimates were proved.

- Matsumoto (1988) $D_\sigma(T;\mathcal{I}) \ll (|\mathcal{I}|+1)(\log\log T)^{-\frac{2\sigma-1}{15}+\epsilon}$
- Harman–Matsumoto (1994) $D_\sigma(T;\mathcal{I}) \ll (|\mathcal{I}|+1)(\log T)^{-\frac{4\sigma-2}{8\sigma+21}+\epsilon}$
- M. (2019) $D_\sigma(T;\mathcal{I}) \ll (|\mathcal{I}|+1)(\log T)^{-\frac{1}{2}+\epsilon}$

The best upper bound known to date has been achieved by Lamzouri–Lester–Radziwiłł (2019).

Discrepancy estimate for $\zeta(\sigma + it)$

Discrepancy estimate by Lamzouri–Lester–Radziwiłł

Let $1/2 < \sigma \leq 1$. Then we have

$$\sup_{\mathcal{I} \subset \mathbb{R}} D_\sigma(T; \mathcal{I}) \ll \begin{cases} (\log T)^{-\sigma} & \text{for } 1/2 < \sigma < 1, \\ (\log T)^{-1}(\log\log T)^2 & \text{for } \sigma = 1. \end{cases}$$

Remark:

The above results were originally obtained as upper bounds for

$$D_\sigma(T; \mathcal{R}) = \left| \frac{1}{T} \operatorname{meas}\left\{ t \in [0, T] \;\middle|\; \log\zeta(\sigma + it) \in \mathcal{R} \right\} - \iint_{\mathcal{R}} \mathcal{M}_\sigma(x + iy)\, \frac{dxdy}{2\pi} \right|,$$

where $\mathcal{R} = [a, b] \times i[c, d]$ and $\mathcal{M}_\sigma : \mathbb{C} \to \mathbb{R}_{\geq 0}$.

Discrepancy estimate for $L(\sigma, f)$

Let $\sigma > 1/2$ and $\mathcal{I} = [a, b] \subset \mathbb{R}$. Then we define

$$\mathscr{D}_\sigma(q; \mathcal{I}) = \left| \frac{1}{\# B_2(q)} \# \{ f \in B_2(q) \mid \log L(\sigma, f) \in \mathcal{I} \} - \int_{\mathcal{I}} \mathscr{M}_\sigma(x) \frac{dx}{\sqrt{2\pi}} \right|.$$

Theorem 1 (M., arXiv:2011.07504)

Let $1/2 < \sigma \leq 1$. Then we have

$$\sup_{\mathcal{I} \subset \mathbb{R}} \mathscr{D}_\sigma(q; \mathcal{I}) \ll \begin{cases} (\log q)^{-\sigma} & \text{for } 1/2 < \sigma < 1, \\ (\log q)^{-1} (\log\log q \ \log\log\log q) & \text{for } \sigma = 1. \end{cases}$$

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A basic inequality

Recall that the proof of Berry–Esseen's theorem was based on the following inequality.

Lemma

Let P_1 and P_2 be two probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Put

$$F_j(t) = P_j((-\infty, t]) \quad \text{and} \quad f_j(u) = \int_{\mathbb{R}} e^{ixu}\, dP_j(x)$$

for $j = 1, 2$. Assume that $F_2(t)$ is differentiable. Then we have

$$\sup_{\mathcal{I} \subset \mathbb{R}} |P_1(\mathcal{I}) - P_2(\mathcal{I})| \leq \frac{4}{\pi} \int_0^R \frac{|f_1(u) - f_2(u)|}{u}\, du + \frac{48K}{\pi} R^{-1}$$

for any $R > 0$, where $K = \sup_{t \in \mathbb{R}} |F_2'(t)|$.

Key propositions

To apply the lemma for the proof of Theorem 1, we consider the characteristic functions

$$f_1(u) = \frac{1}{\#B_2(q)} \sum_{f \in B_2(q)} \exp\left(iu \log L(\sigma, f)\right) \quad \text{and} \quad f_2(u) = \int_{\mathbb{R}} e^{ixu} \mathscr{M}_\sigma(x) \frac{dx}{\sqrt{2\pi}}.$$

Proposition 1

Let $1/2 < \sigma \leq 1$. Then there exist a constant $A_\sigma > 0$ such that

$$f_1(u) = f_2(u) + O\left(\frac{1}{(\log q)^2}\right)$$

holds uniformly for $|u| \leq A_\sigma R_\sigma(q)$, where

$$R_\sigma(q) = \begin{cases} (\log q)^\sigma & \text{for } 1/2 < \sigma < 1, \\ (\log q)(\log\log q \ \log\log\log q)^{-1} & \text{for } \sigma = 1. \end{cases}$$

Key propositions

Proposition 2

Let $1/2 < \sigma \leq 1$. Then the formula

$$f_j(u) = 1 + O(|u|)$$

holds uniformly for $u \in \mathbb{R}$ for each $j = 1, 2$.

The main tools for the proofs of Propositions 1 and 2 are the followings:

- Eichler–Selberg trace formula,
- a zero-density estimate for $L(s, f)$,
- a large sieve inequality for $\lambda_f(n)$.

Proof of Theorem 1

$$\sup_{\mathcal{I} \subset \mathbb{R}} |P_1(\mathcal{I}) - P_2(\mathcal{I})| \leq \frac{4}{\pi} \int_0^R \frac{|f_1(u) - f_2(u)|}{u} \, du + \frac{48K}{\pi} R^{-1} \tag{3.1}$$

We apply (3.1) for the probability measures defined as

$$P_1(A) = \frac{1}{\# B_2(q)} \# \left\{ f \in B_2(q) \;\middle|\; \log L(\sigma, f) \in A \right\} \quad \text{and} \quad P_2(A) = \int_A \mathcal{M}_\sigma(x) \frac{dx}{\sqrt{2\pi}}.$$

Note that $K = \sup_{t \in \mathbb{R}} \frac{\mathcal{M}_\sigma(t)}{\sqrt{2\pi}} \ll 1$. Furthermore, we choose the parameter R as

$$R = A_\sigma R_\sigma(q) = \begin{cases} A_\sigma (\log q)^\sigma & \text{for } 1/2 < \sigma < 1, \\ A_\sigma (\log q)(\log\log q \, \log\log\log q)^{-1} & \text{for } \sigma = 1, \end{cases}$$

where A_σ and $R_\sigma(q)$ are as in Proposition 1.

Put $r = (\log q)^{-2}$. We evaluate the integral in RHS of (3.1) as follows.

$\underline{r < u < R}$ By Proposition 1, we have

$$\int_r^R \frac{|f_1(u) - f_2(u)|}{u} \, du \ll \frac{1}{(\log q)^2} \int_r^R \frac{du}{u} \ll \frac{\log\log q}{(\log q)^2}.$$

$\underline{0 < u < r}$ By Proposition 2, we have

$$\int_0^r \frac{|f_1(u) - f_2(u)|}{u} \, du \ll \int_0^r \, du \ll \frac{1}{(\log q)^2}.$$

Therefore, we deduce from (3.1) the desired upper bound

$$\begin{aligned}\sup_{\mathcal{I} \subset \mathbb{R}} \mathscr{D}_\sigma(q; \mathcal{I}) = \sup_{\mathcal{I} \subset \mathbb{R}} |P_1(\mathcal{I}) - P_2(\mathcal{I})| &\ll \frac{\log\log q}{(\log q)^2} + \frac{1}{(\log q)^2} + \frac{1}{R_\sigma(q)} \\ &\ll \frac{1}{R_\sigma(q)}.\end{aligned}$$

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Bounds for $L(\sigma, f)$

convexity bound	$L(\sigma, f) = O\left(q^{\frac{1-\sigma}{2}+\epsilon}\right)$	for $0 \le \sigma \le 1$
Lindelöf Hypothesis	$L(\sigma, f) = O\left(q^{\epsilon}\right)$	for $1/2 \le \sigma \le 1$
under GRH	$\log L(\sigma, f) = O\left(\frac{(\log q)^{2-2\sigma}}{\log\log q}\right)$	for $1/2 < \sigma < 1$
conjecture	$\log L(\sigma, f) = O\left((\log q)^{1-\sigma+o(1)}\right)$	for $1/2 < \sigma < 1$

Let $1/2 < \sigma < 1$ and $\tau \in \mathbb{R}$. Then we define

$$\Phi_q(\sigma, \tau) = \frac{1}{\# B_2(q)} \# \{ f \in B_2(q) \mid \log L(\sigma, f) > \tau \} .$$

Question: How $\Phi_q(\sigma, \tau)$ behaves when $\tau \approx (\log q)^{1-\sigma+o(1)}$?

Distribution of extreme values of $L(\sigma, f)$

If $\tau \in \mathbb{R}$ is fixed, then we have

$$\Phi_q(\sigma,\tau) \to \Phi(\sigma,\tau) := \int_\tau^\infty \mathscr{M}_\sigma(x)\,\frac{dx}{\sqrt{2\pi}}$$

as $q \to \infty$ by Golubeva's limit theorem. More precisely, we have the following result.

Theorem 2 (M., in preparation)

For $1/2 < \sigma < 1$, there exists a constant $c(\sigma) > 0$ such that

$$\frac{\Phi_q(\sigma,\tau)}{\Phi(\sigma,\tau)} = 1 + O\left(\frac{(\tau\log\tau)^{\frac{\sigma}{1-\sigma}}}{(\log q)^\sigma}\right)$$

holds uniformly in the range $1 \ll \tau \le c(\sigma)(\log q)^{1-\sigma}(\log\log q)^{-1}$.

Distribution of extreme values of $L(\sigma, f)$

Furthermore, $\Phi(\sigma,\tau)$ satisfies the following asymptotic formula.

Theorem 3 (M., in preparation)

Let $1/2 < \sigma < 1$ and $N \in \mathbb{Z}_{\geq 1}$. For $n = 0, \ldots, N-1$, there exist a positive constant $A(\sigma)$ and polynomials $A_{n,\sigma}(x)$ of degree at most n with $A_{0,\sigma}(x) = 1$ such that

$$\Phi(\sigma,\tau) = \exp\left(-A(\sigma)\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}\left\{\sum_{n=0}^{N-1}\frac{A_{n,\sigma}(\log\log\tau)}{(\log\tau)^n} + O\left(\left(\frac{\log\log\tau}{\log\tau}\right)^N\right)\right\}\right)$$

holds if $\tau > 0$ is large enough.

- $A(\sigma)$ and $A_{n,\sigma}(x)$ can be explicitly described by using the function

$$g(u) = \log\left(\frac{2}{\pi}\int_0^\pi \exp(2u\cos\theta)\sin^2\theta\, d\theta\right) = \log(I_1(2u)/u).$$

Distribution of extreme values of $L(\sigma, f)$

Corollary

Let $1/2 < \sigma < 1$ and $N \in \mathbb{Z}_{\geq 1}$. Then there exists a constant $c(\sigma) > 0$ such that

$$\Phi_q(\sigma, \tau) = \exp\left(-A(\sigma)\tau^{\frac{1}{1-\sigma}}(\log \tau)^{\frac{\sigma}{1-\sigma}}\left\{\sum_{n=0}^{N-1} \frac{A_{n,\sigma}(\log\log \tau)}{(\log \tau)^n} + O\left(\left(\frac{\log\log \tau}{\log \tau}\right)^N\right)\right\}\right)$$

holds uniformly in the range $1 \ll \tau \leq c(\sigma)(\log q)^{1-\sigma}(\log\log q)^{-1}$, where $A(\sigma)$ and $A_{n,\sigma}(x)$ are as in Theorem 3.

- When $N = 1$, a similar asymptotic formula for

$$\widetilde{\Phi}_q(\sigma, \tau) = \left(\sum_{f \in B_2(q)} \omega_f\right)^{-1} \sum_{\substack{f \in B_2(q) \\ \log L(\sigma, f) > \tau}} \omega_f$$

was proved by Lamzouri (2011), where $\omega_f := 1/4\pi\langle f, f\rangle$.

Thank you for your kind attention!

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$$\Phi_q(\sigma,\tau) = \exp\left(-A(\sigma)\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}\left\{\sum_{n=0}^{N-1}\frac{A_{n,\sigma}(\log\log\tau)}{(\log\tau)^n} + O\left(\left(\frac{\log\log\tau}{\log\tau}\right)^N\right)\right\}\right)$$

Let $g(u) = \log(I_1(2u)/u)$ as before, and put

$$\mathfrak{a}_0(\sigma) = \int_0^\infty g(y^{-\sigma})\,dy \quad \text{and} \quad \mathfrak{a}_1(\sigma) = \int_0^\infty g(y^{-\sigma})\log y\,dy.$$

Then the constant $A(\sigma)$ is represented as

$$A(\sigma) = (1-\sigma)\left(\frac{1-\sigma}{\sigma}\mathfrak{a}_0(\sigma)\right)^{-\frac{\sigma}{1-\sigma}},$$

and the polynomials $A_{n,\sigma}(x)$ are represented as

$$A_{0,\sigma}(x) = 1, \qquad A_{1,\sigma}(x) = \frac{\sigma}{1-\sigma}\left\{x - \log\left(\frac{1-\sigma}{\sigma}\mathfrak{a}_0(\sigma)\right) + \frac{1-\sigma}{\sigma}\frac{\mathfrak{a}_1(\sigma)}{\mathfrak{a}_0(\sigma)}\right\}, \qquad \ldots.$$