The Impossibility of a Fixed-step Anonymous Extension of the Catching-up Criterion: A Re-examination †

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Abstract

The generalized Pareto axiom, called $\mathcal{N}$-Pareto, is introduced to re-examine the impossibility of a fixed-step anonymous extension of the catching-up criterion for infinite utility streams. $\mathcal{N}$-Pareto postulates that the evaluation is positively sensitive to the utilities of generations whose coalition belongs to a given family $\mathcal{N}$ of sets of generations. Strong Pareto, one of the axioms that the utilitarian and leximin versions of the catching-up criterion satisfy, is a special case of $\mathcal{N}$-Pareto, which is defined by the family $\mathcal{N}$ of all non-empty subsets of the set of all generations. It is shown that $\mathcal{N}$-Pareto is compatible with the axiom of fixed-step anonymity and the axiom of consistency that the utilitarian and leximin versions of the catching-up criterion satisfy if and only if every set of generations in $\mathcal{N}$ consists of more than one generation.

Keywords : Social Choice Theory, Intergenerational Equity, Catching-up Criterion, Fixed-step Anonymity, Pareto Axioms

I Introduction

To give a socially optimal solution to intergenerational economic problems that affect the welfare not only of the present but of future generations (e.g., the choice from alternative programs of greenhouse gas abatement), an evaluation relation that compares relative goodness of alternative solutions is needed. In optimal growth theory, one of the most famous optimality concepts is the catching-up criterion introduced by Atsumi (1965) and von Weizsäcker (1965). This criterion determines an optimal consumption stream by the comparison of partial sums of utilities generated by consumption streams. In social choice theory, Svensson (1980) reformulates the catching-up criterion as a social welfare quasi-ordering (SWQ) on the set of infinite utility streams.[1)] An infinite utility stream is an infinite-dimentional vector which represents utility levels that, we presume, infinitely many generations (listed from the present generation to infinite future generations) will attain under a per-period social state (or a consumption level). The leximin version of the catching-up SWQ is introduced by Asheim

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and Tungodden (2004). It applies the leximin (=lexicographic maximin) principle to truncated utility streams. To distinguish the two versions of catching-up SWQs, we will refer to the one by Svensson (1980) as *utilitarian catching-up* SWQ and the leximin version as *leximin catching-up* SWQ.

Asheim and Tungodden (2004) examines the normative foundation of the utilitarian and leximin catching-up SWQs in terms of some axioms stated for the evaluation of infinite utility streams.[2] An axiom is a formalization of good property that the evaluation should satisfy. By the analysis of Asheim and Tungodden (2004), it is known that the utilitarian and leximin catching-up SWQs satisfy some axioms including the two basic ones: *Strogn Pareto* (**SP**) and *Finite Anonymity* (**FA**).[3] **SP** postulates that the evaluation is positively sensitive to the utilities of generations. **FA** formalizes equal treatment of generations by asserting that two utility streams are equally good if they coincide with each other by exchanging the positions of a finite number of generations. Since the seminal work by Diamond (1965), **FA** is employed by many researchers working on the axiomatic analysis of the evaluation of infinite utility streams.[4] However, the drawback of **FA** is that it cannot conclude the following two streams are equally good:

$$\begin{cases} x = (1, 0, 1, 0, 1, 0, \ldots), \\ y = (0, 1, 0, 1, 0, 1, \ldots). \end{cases}$$

According to our ethical intuition, x and y should be declared equally good. The stronger anonymity axiom that asserts such an evaluation is proposed by Lauwers (1997b) under the name *Fixed-step Anonymity* (**SA**). **SA** asserts that two utility streams are equally good if they coincide with each other by exchanging the positions of a fixed length of contiguous generations. In the example above, $\boldsymbol{x}$ and $\boldsymbol{y}$ coincides by exchanging the positions of two contiguous generations. Although the equal treatment of generations formalized by **SA** is intuitively appealing, Banerjee (2006) shows that any extension of the utilitarian catching-up SWQs (thus, also the utilitarian catching-up SWQs itself) violates it. Further, using a Weak Dominace (**WD**), which is a Pareto axiom weaker than **SP**, Kamaga and Kojima (2010) show that **WD**, **SA** and the consistency axiom called *Restricted Strong Preference Consistency* (**RSPC**) are incompatible. **RSPC** formalizes the consistency property that the utilitarian and leximin catching-up SWQs satisfy. Hence, their result implies that any extension of the leximin SWQ also violates **SA**.

In this paper, we re-examine the impossibility of a fixed-step anonymous extension of the utilitarian and leximin catching-up SWQs. In particular, we re-examine this with a focus on Pareto axioms that these SWQs satisfy. For this purpose, we introduce the generalized Pareto axiom called $\mathcal{N}$-Pareto ($\mathcal{N}$**P**). $\mathcal{N}$**P** postulates that the evaluation is positively sensitive to the utilities of generations whose coalition belongs to a given family $\mathcal{N}$ of sets of generations. Accoding to the choice of $\mathcal{N}$, $\mathcal{N}$**P** represents several Pareto axioms. For example, **SP** corresponds to $\mathcal{N}$**P** defined by the family $\mathcal{N}$ of all non-empty subsets of the set of all generations. We examine which families $\mathcal{N}$ of sets of generations defines $\mathcal{N}$**P** that is compatible with the existence of an SWQ that satisfies **SA** and **RSPC**. In Theorem 1, It is shown that $\mathcal{N}$**P** is compatible with the existence of an SWQ that satisfies the axioms if and only if every set of generations in $\mathcal{N}$ consists of more than one generation. This result strengthens the above-mentioned impossibility obtained by Kamaga and Kojima (2010) and another result of them that shows the existence of an SWQ that satisfies Weak Pareto (**WP**), **SA**, and **RSPC**.

The rest of the paper is organized as follows. In Sect. II, we set up the framework of the analysis, and the definitions of the utilitarian and leximin catching-up SWQs are presented. The existing axioms we consider are also presented. In Sect. III, we introduce $\mathcal{N}$-Pareto, and Theorem 1 is established. Sect. IV concludes the study.

II Preliminary

1. Notation and definitions

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{N}$ be the set of all positive integers. A utility stream is denoted by $\boldsymbol{x} = (x_1, x_2, \ldots)$ where x_i is interpreted as the utility level of the i-th generation for all $i \in \mathbb{N}$. Let X denote a set of utility streams. Except for some results that we will mention explicitly, we let $X = \mathbb{R}^{\mathbb{N}}$. For all $\boldsymbol{x} \in X$ and all $n \in \mathbb{N}$, we write $\boldsymbol{x}^{-n} = (x_1, \ldots, x_n)$ and $\boldsymbol{x}^{+n} = (x_{n+1}, x_{n+2}, \ldots)$. We will refer to $\boldsymbol{x}^{+n}$ as *tail* of $\boldsymbol{x}$. For all $\boldsymbol{x} \in X$ and all $n \in \mathbb{N}$, $(x^{-n}_{(1)}, \ldots, x^{-n}_{(n)})$ denotes a rank-ordered rearrangement of $\boldsymbol{x}^{-n}$ such that $x^{-n}_{(1)} \leq \cdots \leq x^{-n}_{(n)}$, ties being broken arbitrarily.

Negation of a statement is indicated by the symbol $\neg$. Our notation for vector inequalities on X is as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$, (i) $\boldsymbol{x} \geq \boldsymbol{y}$ if $x_i \geq y_i$ for all $i \in \mathbb{N}$, (ii) $\boldsymbol{x} > \boldsymbol{y}$ if $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$, and (iii) $\boldsymbol{x} \gg \boldsymbol{y}$ if $x_i > y_i$ for all $i \in \mathbb{N}$. We also use the same notation for finite-horizon vectors.

Given a set A, we denote the set of all subsets of A by 2^A. Let $\varnothing$ denote the empty set. Given two sets A and B, we write $A \subseteq B$ to mean A is a subset of B; and $A \subset B$ to mean $A \subseteq B$ and $A \neq B$. For any set A, $|A|$ is the cardinality of A and we write $|A| = \infty$ if $|A|$ is (countably) infinite.

A binary relation $\succsim$ on X is a subset of $X \times X$. For convenience, the fact that $(\boldsymbol{x}, \boldsymbol{y}) \in \succsim$ will be symbolized by $\boldsymbol{x} \succsim \boldsymbol{y}$. We interpret binary relations on X as social goodness relations, i.e., $\boldsymbol{x} \succsim \boldsymbol{y}$ means that, from the viewpoint of the society, $\boldsymbol{x}$ is at least as good as $\boldsymbol{y}$.[5] The asymmetric part of $\succsim$ is denoted by $\succ$ and the symmetric part by $\sim$, i.e., $\boldsymbol{x} \succ \boldsymbol{y}$ if and only if $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\neg(\boldsymbol{y} \succsim \boldsymbol{x})$, and $\boldsymbol{x} \sim \boldsymbol{y}$ if and only if $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\boldsymbol{y} \succsim \boldsymbol{x}$. The interpretation of $\succ$ and $\sim$ is such that $\boldsymbol{x} \succ \boldsymbol{y}$ (resp. $\boldsymbol{x} \sim \boldsymbol{y}$) means that $\boldsymbol{x}$ is better than (resp. as good as) $\boldsymbol{y}$.

A binary relation $\succsim$ on X is (i) *reflexive* if, for all $\boldsymbol{x} \in X$, $\boldsymbol{x} \succsim \boldsymbol{x}$, (ii) *transitive* if, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$, $\boldsymbol{x} \succsim \boldsymbol{z}$ holds whenever $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\boldsymbol{y} \succsim \boldsymbol{z}$, and (iii) *complete* if, for all $\boldsymbol{x}, \boldsymbol{y} \in X$ with $\boldsymbol{x} \neq \boldsymbol{y}$, $\boldsymbol{x} \succsim \boldsymbol{y}$ or $\boldsymbol{y} \succsim \boldsymbol{x}$. A social welfare quasi-ordering (SWQ) is a reflexive and transitive binary relation. A social welfare ordering (SWO) is a complete SWQ.

A binary relation $\succsim_A$ on X is a *subrelation* of a binary relation $\succsim_B$ if, for all $\boldsymbol{x}, \boldsymbol{y} \in X$, (i) $\boldsymbol{x} \succsim_A \boldsymbol{y}$ implies $\boldsymbol{x} \succsim_B \boldsymbol{y}$ and (ii) $\boldsymbol{x} \succ_A \boldsymbol{y}$ implies $\boldsymbol{x} \succ_B \boldsymbol{y}$. Conversely, a binary relation $\succsim_A$ is an *extension* of a binary relation $\succsim_B$ if $\succsim_B$ is a subrelation of $\succsim_A$.

2. Catching-up criterion and Pareto and consistency axioms

Based on the catching-up criterion due to Atsumi (1965) and von Weizsäcker (1965), Svensson (1980) formulates the *utilitarian catching-up* SWQ on X in his analysis of the existence of an ethical SWO on X.[6] Further, the leximin version, which we call the *leximin catching-up* SWQ, is introduced by Asheim and Tungodden (2004).[7] These SWQs compare utility streams by applying the finite-horizon utilitarian SWO and the finite-horizon leximin SWO, respectively, for truncated streams (Definitions are given below). For all $n \in \mathbb{N}$, let $\succsim^n_U$ denote the finite-horizon utilitarian SWO on $\mathbb{R}^n$: for all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$,

$$\boldsymbol{x}^{-n} \succsim^n_U \boldsymbol{y}^{-n} \Leftrightarrow \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} y_i,$$

and let $\succsim^n_L$ be the finite-horizon leximin SWO on $\mathbb{R}^n$: for all $\boldsymbol{x}^{-n}, \boldsymbol{y}^{-n} \in \mathbb{R}^n$,

$$\boldsymbol{x}^{-n} \succsim_L^n \boldsymbol{y}^{-n} \Leftrightarrow \begin{cases} (x_1, \ldots, x_n) \text{ is a permutation of } (y_1, \ldots, y_n) \\ \text{or there exists } m \in \{1, \ldots, n\} \text{ such that} \\ x_{(i)} = y_{(i)} \text{ for all } i < m \text{ and } x_{(m)} > y_{(m)}. \end{cases}$$

The utilitarian catching-up SWQ compares utility streams according to the domination in terms of the utilitarian evaluation of trancated streams. The utilitarian catching-up SWQ is defined as the following binary relation $\succsim_{UC}$ on X: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{UC} \boldsymbol{y} \Leftrightarrow \text{there exists } \bar{n} \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-n} \succsim_U^n \boldsymbol{y}^{-n} \text{ for all } n \geq \bar{n}.$$

The leximin catching-up SWQ compares utility streams in a similar manner to the utilitarian catching-up SWQ. It uses the finite-horizon leximin SWO instead of the finite-horizon utilitarian SWO. The leximin catching-up SWQ is defined as the following binary relation $\succsim_{LC}$ on X: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim_{LC} \boldsymbol{y} \Leftrightarrow \text{there exists } \bar{n} \in \mathbb{N} \text{ such that } \boldsymbol{x}^{-n} \succsim_L^n \boldsymbol{y}^{-n} \text{ for all } n \geq \bar{n}.$$

Let us list the Pareto and consistency axioms that the utilitarian and leximin catching-up SWQs satisfy. The first one is Strong Pareto, which has been widely used in the literature since the siminal work by Diamond (1965). It postulates positive sensitivity of the evaluation to utilities of generations.

Strong Pareto (SP): For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if $\boldsymbol{x} > \boldsymbol{y}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.

Two weakenings of **SP** are employed in the literature.

Weak Pareto (WP): For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if $\boldsymbol{x} \gg \boldsymbol{y}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.
Weak Dominance (WD): For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if there exists $i \in \mathbb{N}$ such that $x_i > y_i$ and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.

WD is proposed by Basu and Mitra (2003). There is no logical relationship between **WP** and **WD**.

The next axiom is Restricted Strong Preference Consistency, which is introduced in Kamaga and Kojima (2010).[8] It asserts that the evaluation of two utility streams must be consistent with the evaluations that we conclude for those streams whose tails are replaced with a common one.

Restricted Strong Preference Consistency (RSPC): For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if for all $\boldsymbol{z} \in X$, $(\boldsymbol{x}^{-n}, \boldsymbol{z}^{+n}) \succsim (\boldsymbol{y}^{-n}, \boldsymbol{z}^{+n})$ for all $n \in \mathbb{N}$, and for all $n \in \mathbb{N}$, there exists $n' \geq n$ such that $(\boldsymbol{x}^{-n'}, \boldsymbol{z}^{+n'}) \succ (\boldsymbol{y}^{-n'}, \boldsymbol{z}^{+n'})$, then $\boldsymbol{x} \succ \boldsymbol{y}$.

It is known that any extension of the utilitarian catching-up SWQ satisfies **SP** (thus, **WP** and **WD**) and **RSPC** and the same is true for any extension of the leximin catching-up SWQ. On this, see Asheim and Tungodden (2004) and Kamaga and Kojima (2010).[9]

3. Permutations and anonymity axioms

A permutation on $\mathbb{N}$ is a bijection on $\mathbb{N}$. Following Mitra and Basu (2007), we represent a permutation on $\mathbb{N}$ by a permutation matrix. A permutation matrix is an infinite matrix $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}}$ such that

(i) for all $i \in \mathbb{N}$, there exists $j(i) \in \mathbb{N}$ such that $p_{ij(i)} = 1$ and $p_{ij} = 0$ for all $j \neq j(i)$;
(ii) for all $j \in \mathbb{N}$, there exists $i(j) \in \mathbb{N}$ such that $p_{i(j)j} = 1$ and $p_{ij} = 0$ for all $i \neq i(j)$.

Given a permutation π on $\mathbb{N}$, there uniquely exists a permutation matrix $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}}$ such that, for all $x \in X$, $(x_{\pi(1)}, x_{\pi(2)}, \ldots) = \boldsymbol{P}x$, where the product $\boldsymbol{P}x = (Px_1, Px_2, \ldots)$ is defined by $Px_i = \Sigma_{k \in \mathbb{N}} p_{ik} x_k$ for all $i \in \mathbb{N}$.[10] Note that $(x_{\pi(1)}, x_{\pi(2)}, \ldots) = \boldsymbol{P}x$ for all $x \in X$ if and only if $p_{ij} = 1$ for all $i \in \mathbb{N}$ and all $j = \pi(i)$.

Let $\mathcal{P}$ denote the set of all permutation matrices. For any $\boldsymbol{P} \in \mathcal{P}$, let $\boldsymbol{P}'$ be the inverse of $\boldsymbol{P}$ satisfying $\boldsymbol{P}'\boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}' = \boldsymbol{I}$, where $\boldsymbol{I}$ is the infinite identity matrix.[11] For all $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}} \in \mathcal{P}$ and all $n \in \mathbb{N}$, $\boldsymbol{P}(n)$ denotes the $n \times n$ matrix $(p_{ij})_{i,j \in \{1,\ldots,n\}}$.

An axiom of anonymity is defined for a given $\mathcal{Q} \subseteq \mathcal{P}$. It formalizes equal treatment of generations by asserting that two streams that coincide with each other by applying a permutation $\boldsymbol{P} \in \mathcal{Q}$ are equally good. For any $\mathcal{Q} \subseteq \mathcal{P}$, $\mathcal{Q}$-Anonymity is defined as follows.

$\mathcal{Q}$-**Anonymity**: For all $x \in X$ and all $\boldsymbol{P} \in \mathcal{Q}$, $\boldsymbol{P}x \sim x$.

The strongest anonymity is defined by $\mathcal{Q} = \mathcal{P}$. It is known, however, that there is no binary relation on X that satisfies $\mathcal{P}$-Anonymity and **SP** (Lauwers, 1997a; Liedekerke, 1995).[12] The following two subclasses of $\mathcal{P}$ have received much attention in the literature:

$$\mathcal{F} = \{\boldsymbol{P} \in \mathcal{P} : \text{there exists } n \in \mathbb{N} \text{ such that } p_{ii} = 1 \text{ for all } i > n\},$$

$$\mathcal{S} = \left\{\boldsymbol{P} \in \mathcal{P} : \begin{array}{l}\text{there exists } k \in \mathbb{N} \text{ such that, for each } n \in \mathbb{N},\\ \boldsymbol{P}(nk) \text{ is a finite-dimensional permutation matrix}\end{array}\right\}.$$

$\mathcal{F}$ is the set of all *finite permutations*, and $\mathcal{F}$-Anonymity is called Finite Anonymity (**FA**). **FA** is first presented by Diamind (1965). $\mathcal{S}$ is the set of all *fixed-step permutations*, and $\mathcal{S}$-Anonymity is called Fixed-step Anonymity (**SA**). **SA** is proposed by Lauwers (1997b). Since $\mathcal{F} \subset \mathcal{S}$, **SA** implies **FA**.

Any extension of the utilitarian catching-up SWQ and any extension of the leximin catching-up SWQ satisfy **FA** (Asheim and Tungodden, 2004; Basu and Mitra, 2007). However, they violate **SA** since they declare x is better than y for the streams we presented in Sect. I.

I close this section with an expositional remark regarding **SA**. SWQs that satisfy **SP** and **SA** (thus, **FA**) have been presented by many recent works.[13] However, **SA** is *not* a *maximal* anonymity that is compatible with the existence of a strongly Paretian SWQ. Nevertheless, **SA** received much attention in the literature since, as proved by Lauwers (2012), a maximal anonymity that is compatible with the existence of a strongly Paretian SWQ involves the use of non-constructive mathematics (such like Axiom of Choice or the existence of a free ultrafilter) and its explicit definition can never be obtained. That **SA** is explicitly defined is another merit of using **SA** as a strengthening of **FA**. For the reader who is interested in the issue of maximal anonymity compatible with the existence of a strongly Paretian SWQ, I present, in Appendix, a short review of this issue and a demonstration of the existence of $\mathcal{Q} \subseteq \mathcal{P}$ that defines the Pareto-compatible maximal anonymity using Zorn's lemma.

III Generalized Pareto axiom and compatibility with SA and RSPC

In this section, we re-examine the impossibility of a fixed-step anonymous extensions of the utilitarian and leximin catching-up SWQs with a focus on Pareto axioms. Banerjee (2006) is the first who pointed out that any extension of the utilitarian catching-up SWQ violates **SA**. Kamaga and Kojima (2010) generalize this impossibility by showing that there is no SWQ on X that satisfies **WD**, **SA**, and **RSPC** as soon X includes a binary domain (e.g., $\{0, 1\}^{\mathbb{N}}$). Their result is stated as follows.

Proposition 1 (Kamaga and Kojima, 2010, Proposition 3). *Let $X \supseteq \{v, w\}^{\mathbb{N}}$ with $v, w \in \mathbb{R}$ and $v \neq w$. There is no SWQ on X that satisfies **WD**, **SA**, and **RSPC**.*

Recall that any extension of the leximin catching-up SWQ satisfies **SP** (thus, **WD**) and **RSPC**. Thus, by Proposition 1, any extension of the leximin catching-up SWQ violates **SA**.

As we have noted in Sect. II 2., **WP** is another weakening of **SP** other than **WD**. Kamaga and Kojima (2010) show that if we replace **WD** with **WP** in Proposition 1, there exists an SWQ that satisfies all three axioms.

Proposition 2 (Kamaga and Kojima, 2010, Proposition 4). *There exists an SWQ on X that satisfies **WP**, **SA**, and **RSPC**.*

By Propositions 1 and 2, not all but some implications of **SP** is essential for the impossibility of a fixed-step anonymous extention of the utilitarian and leximin catching-up SWQs. To examine what implication is essential to the problem, we consider the generalized Pareto axiom which we call $\mathcal{N}$-Pareto. Given a set $\mathcal{N}$ of non-empty subsets of $\mathbb{N}$, i.e., $\mathcal{N} \subseteq 2^{\mathbb{N}} \setminus \{\varnothing\}$, $\mathcal{N}$-Pareto associated with $\mathcal{N}$ postulates that the evaluation is positively sensitive to utilities of generations whose coalition belongs to $\mathcal{N}$. For an arbitrary $\mathcal{N} \subseteq 2^{\mathbb{N}} \setminus \{\varnothing\}$, $\mathcal{N}$-Pareto is define as follows.

$\mathcal{N}$-Pareto ($\mathcal{N}$P): For all $\boldsymbol{x}, \boldsymbol{y} \in X$, if $\boldsymbol{x} \geq \boldsymbol{y}$ and $\{i \in \mathbb{N} : x_i > y_i\} \in \mathcal{N}$, then $\boldsymbol{x} \succ \boldsymbol{y}$.

By definition, for any subsets $\mathcal{N}, \mathcal{N}' \subseteq 2^{\mathbb{N}} \setminus \{\varnothing\}$, if $\mathcal{N} \subseteq \mathcal{N}'$ then $\mathcal{N}'$**P** implies $\mathcal{N}$**P**. According to the choice of $\mathcal{N}$, $\mathcal{N}$**P** represents various Pareto axioms. $\mathcal{N}$**P** is equivalent, respectively, to (i) **SP**, (ii) **WP**, (iii) **WD**, and (iv) Strong Monotonicity for Infinite Generations proposed by Sakai (2006) if it is defined by[14)]

(i) $\mathcal{N} = 2^{\mathbb{N}} \setminus \{\varnothing\}$,
(ii) $\mathcal{N} = \{\mathbb{N}\}$,
(iii) $\mathcal{N} = \{N \in 2^{\mathbb{N}} : |N| = 1\}$,
(iv) $\mathcal{N} = \{N \in 2^{\mathbb{N}} : |N| = \infty\}$.

The following theorem establishes a characterization of $\mathcal{N}$**P** that is compatible with the existence of an SWQ that satisfies **SA** and **RSPC**. It shows that $\mathcal{N}$**P** is compatible with the existence of an SWQ that satisfies these axioms if and only if $\mathcal{N}$ does not contain any singleton sets of generations.

Theorem 1. *Let $\mathcal{N} \subseteq 2^{\mathbb{N}} \setminus \{\varnothing\}$. There exists an SWQ on X that satisfies $\mathcal{N}$**P**, **SA**, and **RSPC** if and only if $|N| > 1$ for all $N \in \mathcal{N}$.*

Proof. [Only-if-part] Suppose that there exists an SWQ $\succsim$ on X that satisfies $\mathcal{N}$**P**, **SA**, and **RSPC**. The proof is done by contradiction. By way of contradiction, suppose that there exists $n \in \mathbb{N}$ such that $\{n\} \in \mathcal{N}$. We first show that $\succsim$ satisfies **WD**. Let $\boldsymbol{x}, \boldsymbol{y} \in X$ and suppose that there exists $n' \in \mathbb{N}$ such that

$$x_{n'} > y_{n'} \text{ and } x_i = y_i \text{ for all } i \in \mathbb{N} \setminus \{n'\}.$$

Consider $\boldsymbol{w}, \boldsymbol{z} \in X$ such that

$$\begin{cases} w_n = x_{n'}, w_{n'} = x_n, \text{ and } w_i = x_i \text{ for all } i \in \mathbb{N}\setminus\{n, n'\}; \\ z_n = y_{n'}, z_{n'} = y_n, \text{ and } z_i = y_i \text{ for all } i \in \mathbb{N}\setminus\{n, n'\}. \end{cases}$$

By $\mathcal{N}$**P**, $\boldsymbol{w} \succ \boldsymbol{z}$. By **SA**, $\boldsymbol{x} \sim \boldsymbol{w}$ and $\boldsymbol{z} \sim \boldsymbol{y}$. Since $\succsim$ is transitive, $\boldsymbol{x} \succ \boldsymbol{y}$. Thus, $\succsim$ satisfies **WD**. However, by Proposition 1, this is a contradiction to that $\succsim$ satisfies **SA** and **RSPC**. Thus, there is no $n \in \mathbb{N}$ such that $\{n\} \in \mathcal{N}$. This implies that $|N| > 1$ for all $N \in \mathcal{N}$.

[If-part] Suppose that $|N| > 1$ for all $N \in \mathcal{N}$. Define the binary relation $\succsim^*$ as follows: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \succsim^* \boldsymbol{y} \Leftrightarrow \text{there exists } \boldsymbol{P} \in \mathcal{S} \text{ such that} \begin{cases} \text{(i) } \boldsymbol{Px} \geq \boldsymbol{y} \text{ and } |\{i \in \mathbb{N} : (\boldsymbol{Px})_i > y_i\}| > 1 \\ \text{or (ii) } \boldsymbol{Px} = \boldsymbol{y}. \end{cases} \tag{1}$$

We show that $\succsim^*$ is an SWQ. Since $\boldsymbol{I} \in \mathcal{S}$, $\succsim^*$ is reflexive. To verify that $\succsim^*$ is transitive, let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$ and suppose that $\boldsymbol{x} \succsim^* \boldsymbol{y}$ and $\boldsymbol{y} \succsim^* \boldsymbol{z}$. By (1), there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{S}$ such that

$$[\boldsymbol{Px} \geq \boldsymbol{y} \text{ and } |\{i \in \mathbb{N} : (\boldsymbol{Px})_i > y_i\}| > 1] \text{ or } \boldsymbol{Px} = \boldsymbol{y},$$

and

$$[\boldsymbol{Qy} \geq \boldsymbol{z} \text{ and } |\{i \in \mathbb{N} : (\boldsymbol{Qy})_i > z_i\}| > 1] \text{ or } \boldsymbol{Qy} = \boldsymbol{z},$$

Then, we obtain

$$[\boldsymbol{QPx} \geq \boldsymbol{z} \text{ and } |\{i \in \mathbb{N} : (\boldsymbol{QPx})_i > z_i\}| > 1] \text{ or } \boldsymbol{QPy} = \boldsymbol{z},$$

since, for all $\boldsymbol{x}, \boldsymbol{y} \in X$ and all $\boldsymbol{P} \in \mathcal{P}$,

$$\boldsymbol{x} \geq \boldsymbol{y} \Leftrightarrow \boldsymbol{Px} \geq \boldsymbol{Py}$$

and, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$ satisfying $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{y} \geq \boldsymbol{z}$,

$$\{i \in \mathbb{N} : x_i > z_i\} = \{i \in \mathbb{N} : x_i > y_i\} \cup \{i \in \mathbb{N} : y_i > z_i\}.$$

Since $\boldsymbol{QP} \in \mathcal{S}$, we obtain $\boldsymbol{x} \succsim \boldsymbol{z}$ by (1).

Next, we show that $\succsim^*$ satisfies $\mathcal{N}$**P**, **SA**, and **RSPC**. To this end, we prove the following equivalence assertions: for all $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\begin{cases} \boldsymbol{x} \succ^* \boldsymbol{y} \Leftrightarrow \text{there exists } \boldsymbol{P} \in \mathcal{S} \text{ such that } \boldsymbol{Px} \geq \boldsymbol{y} \text{ and } |\{i \in \mathbb{N} : (\boldsymbol{Px})_i > y_i\}| > 1; & \text{(2a)} \\ \boldsymbol{x} \sim^* \boldsymbol{y} \Leftrightarrow \text{there exists } \boldsymbol{P} \in \mathcal{S} \text{ such that } \boldsymbol{Px} = \boldsymbol{y}. & \text{(2b)} \end{cases}$$

We first prove the if-part of (2a). Let $\boldsymbol{x}, \boldsymbol{y} \in X$ and suppose that there exists $\boldsymbol{P} \in \mathcal{S}$ such that

$$\boldsymbol{Px} \geq \boldsymbol{y} \text{ and } |\{i \in \mathbb{N} : (\boldsymbol{Px})_i > y_i\}| > 1.$$

By (1), $\boldsymbol{x} \succsim^* \boldsymbol{y}$. We show, by contradiction, that $\neg(\boldsymbol{y} \succsim^* \boldsymbol{x})$. Suppose that $\boldsymbol{y} \succsim^* \boldsymbol{x}$. By (1), there exists $\boldsymbol{Q} \in \mathcal{S}$ such that $\boldsymbol{Qy} \geq \boldsymbol{x}$. Then, we obtain $\boldsymbol{PQy} \geq \boldsymbol{Px} > \boldsymbol{y}$, which contradicts to that $\boldsymbol{PQ} \in \mathcal{S}$. Thus, $\neg(\boldsymbol{y} \succsim^* \boldsymbol{x})$. Next, we prove the only-if-part of (2a). Let $\boldsymbol{x}, \boldsymbol{y} \in X$ and suppose that $\boldsymbol{x} \succ^* \boldsymbol{y}$. By (1), there exists $\boldsymbol{P} \in \mathcal{S}$ such that

$$[\boldsymbol{Px} \geq \boldsymbol{y} \text{ and } |\{i \in \mathbb{N} : (\boldsymbol{Px})_i > y_i\}| > 1] \text{ or } \boldsymbol{Px} = \boldsymbol{y}.$$

Suppose that $\boldsymbol{Px} = \boldsymbol{y}$ holds. Then, $(\boldsymbol{x} =)\boldsymbol{P}'\boldsymbol{Px} = \boldsymbol{P}'\boldsymbol{y}$. Since $\boldsymbol{P}' \in \mathcal{S}$, we obtain $\boldsymbol{y} \succsim^* \boldsymbol{x}$ by (1). This is a contradiction to $\boldsymbol{x} \succ^* \boldsymbol{y}$. Thus, $\boldsymbol{Px} \geq \boldsymbol{y}$ and $|\{i \in \mathbb{N} : (\boldsymbol{Px})_i > y_i\}| > 1$ hold.

Next, we prove (2b). To prove the if-part of (2b), let $\boldsymbol{x}, \boldsymbol{y} \in X$ and suppose that there exists $\boldsymbol{P} \in \mathcal{S}$ such that $\boldsymbol{Px} = \boldsymbol{y}$. By (1), $\boldsymbol{x} \succsim^* \boldsymbol{y}$. Further, since $\boldsymbol{P}' \in \mathcal{S}$ and $(\boldsymbol{x} =)\boldsymbol{P}'\boldsymbol{Px} = \boldsymbol{P}'\boldsymbol{y}$, we obtain $\boldsymbol{y} \succsim^* \boldsymbol{x}$ by (1). Thus, $\boldsymbol{x} \sim^* \boldsymbol{y}$. Finally, we prove the only-if-part of (2b). Let $\boldsymbol{x}, \boldsymbol{y} \in X$ and suppose that $\boldsymbol{x} \sim^* \boldsymbol{y}$. By (1), there exists $\boldsymbol{P} \in \mathcal{S}$ such that

$$[\boldsymbol{Px} \geq \boldsymbol{y} \text{ and } |\{i \in \mathbb{N} : (\boldsymbol{Px})_i > y_i\}| > 1] \text{ or } \boldsymbol{Px} = \boldsymbol{y}.$$

If $\boldsymbol{Px} \geq \boldsymbol{y}$ and $|\{i \in \mathbb{N} : (\boldsymbol{Px})_i > y_i\}| > 1$ hold, then we obtain $\boldsymbol{x} \succ^* \boldsymbol{y}$ by (2a). This is a contradiction to $\boldsymbol{x} \sim^* \boldsymbol{y}$. Thus, $\boldsymbol{Px} = \boldsymbol{y}$ holds.

We now show that $\succsim^*$ satisfies $\mathcal{N}$**P**, **SA**, and **RSPC**. Since $\boldsymbol{I} \in \mathcal{S}$, it follows from (2a) that $\succsim^*$ satisfies $\mathcal{N}$**P**. Further, by (2b), $\succsim^*$ satisfies **SA**. Finally, we verify that $\succsim^*$ satisfies **RSPC** by showing that the premise of **RSPC** is never satisfied by $\succsim^*$. By (2a) and (2b), $(\boldsymbol{x}^{-1}, \boldsymbol{z}^{+1}) \succsim^* (\boldsymbol{y}^{-1}, \boldsymbol{z}^{+1})$ implies

$$x_1 = y_1.$$

Further, for all $n \in \mathbb{N}$, if $\boldsymbol{x}^{-n} = \boldsymbol{y}^{-n}$ and $(\boldsymbol{x}^{-(n+1)}, \boldsymbol{z}^{+(n+1)}) \succsim^* (\boldsymbol{y}^{-(n+1)}, \boldsymbol{z}^{+(n+1)})$, then

$$x_{n+1} = y_{n+1}.$$

Therefore, if $(\boldsymbol{x}^{-n}, \boldsymbol{z}^{+n}) \succsim^* (\boldsymbol{y}^{-n}, \boldsymbol{z}^{+n})$ for all $n \in \mathbb{N}$, then

$$\boldsymbol{x} = \boldsymbol{y},$$

and, by (2b), we obtain that, for all $n \in \mathbb{N}$,

$$(\boldsymbol{x}^{-n}, \boldsymbol{z}^{+n}) \sim^* (\boldsymbol{y}^{-n}, \boldsymbol{z}^{+n}).$$

This means that the premise of **RSPC** is never satisfied by $\succsim^*$. Thus, $\succsim^*$ satisfies **RSPC**. □

By Theorem 1, $\mathcal{N}$**P** defined by

$$\mathcal{N} = \{N \in 2^{\mathbb{N}} : |N| > 1\}$$

is the strongest Pareto axiom that is compatible with the existence of an SWQ satisfying **SA** and **RSPC**. Since

$$\{N \in 2^{\mathbb{N}} : |N| = \infty\} \subset \{N \in 2^{\mathbb{N}} : |N| > 1\},$$

Strong Monotonicity for Infinite Generations is compatible with the existence of an SWQ that satisfies **SA** and **RSPC**.

As shown in the proof, **RSPC** is vacuously true for $\succsim^*$. Thus, $\succsim^*$ is not a satisfactory resolution to the impossibility in Proposition 1. This is a shortcoming of our proof. We leave it as an open question whether Theorem 1 can be proved by an SWQ for which **RSPC** represents a substantial consistency-property.

IV Conclusion

In this paper, we re-examined the impossibility of a fixed-step anonymous extension of the utilitarian and leximin catching-up SWQs with a focus on Paretian properties of these SWQs. To this end, we presented the generalized Pareto axiom called $\mathcal{N}$-Pareto. Using $\mathcal{N}$-Pareto, we showed that, among the implications of the Paretian property of the utilitarian and leximin catching-up SWQs, the positive sensitivity to the utility of a single generation is essential to the impossibility of a fixed-step anonymous extension of these SWQs.

An interesting issue to be addressed is to examine which SWQs $\succsim$ are permissible if we retain **SA** and **RSPC** and require the utilitarian (or leximin) catching-up SWQ to be included in the sense of set inclusion, i.e., $\succsim_{UC} \subseteq \succsim$ (or $\succsim_{LC} \subseteq \succsim$). It follows from Theorem 1 that $\mathcal{N}$-Pareto such SWQs $\succsim$ satisfy must be defined by a family $\mathcal{N}$ that excludes any singleton sets of generations. We leave this issue for future research.

Appendix

As I mentioned in Sect. II 3., **SA** is not a maximal anonymity that is compatible with the existence of an SWQ that satisfies **SP**. Mitra and Basu (2007) provides a necessary and sufficient condition for a set $\mathcal{Q}$ of permutations to define $\mathcal{Q}$-Anonymity that is compatible with the existence of an SWQ satisfying **SP**. They show that a set $\mathcal{Q}$ of permutations defines $\mathcal{Q}$-Anonymity that is compatible with the existence of an SWQ satisfying **SP** if and only if $\mathcal{Q}$ is a *group* of *cyclic* permutations. The term "group" means an algebraic structure. In what follows, we present the definitions of group structure and cyclicity of a permutation.

For any $\mathcal{Q} \subseteq \mathcal{P}$, $\mathcal{Q}$ is a *group* with respect to the matrix multiplication if it satisfies the following properties:[15)]

Property 1 (closure): For all $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q}$, $\boldsymbol{PQ} \in \mathcal{Q}$.
Property 2 (existence of the unit element): For all $\boldsymbol{P} \in \mathcal{Q}$, there exists $\boldsymbol{Q} \in \mathcal{Q}$ such that $\boldsymbol{PQ} = \boldsymbol{QP} = \boldsymbol{P}$.
Property 3 (existence of the inverse element): For all $\boldsymbol{P} \in \mathcal{Q}$, there exists $\boldsymbol{Q} \in \mathcal{Q}$ such that $\boldsymbol{PQ} = \boldsymbol{QP} = \boldsymbol{I}$.

Since $\mathcal{Q}$ is a set of permutations, the permutation $\boldsymbol{Q}$ in Property 2 must be $\boldsymbol{I}$ and the permutation $\boldsymbol{Q}$ in Property 3 must be $\boldsymbol{P}'$.

Let $\boldsymbol{e}^i$ be the stream in X such that 1 in the i-th place and 0 elsewhere, i.e. the i-th unit vector in X. A permutation $\boldsymbol{P} \in \mathcal{P}$ is *cyclic* if, for any $i \in \mathbb{N}$, there exists $n(i) \in \mathbb{N}$ such that $\boldsymbol{P}^{n(i)}\boldsymbol{e}^i = \boldsymbol{e}^i$, where $\boldsymbol{P}^{n(i)}$ denotes the $n(i)$ times iterated multiplication of $\boldsymbol{P}$. The set of all cyclic permutations is denoted by $\mathcal{C}$.

Since permutations that exchange the positions of variable lengths of contiguous generations are cyclic, $\mathcal{S} \subset \mathcal{C}$.[16)] Further, it can be checked that $\mathcal{C} \subset \mathcal{P}$ by $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}} \in \mathcal{P}$ such that (i) $p_{21} = 1$, (ii) $p_{ij} = 1$ for all even j and $i = j + 2$, and (iii) $p_{ij} = 1$ for all odd j with $j \neq 1$ and $i = j - 2$, i.e., the permutation that shifts 1 to 2 and even n to $n + 2$ and shifts backward odd n ($\neq 1$) to $n - 2$). Since $\boldsymbol{P} \notin \mathcal{C}$, $\mathcal{C} \subset \mathcal{P}$. This permutation is the one used by Lauwers (1997a) to prove the incompatibility of **SP** and $\mathcal{Q}$-Anonymity defined by $\mathcal{Q} = \mathcal{P}$. By using $\boldsymbol{P}$, we can construct $\boldsymbol{Px} = (1, 1, 1, 0, 1, 0, \ldots)$ from $\boldsymbol{x} = (1, 0, 1, 0, 1, 0, \ldots)$. By **SP**, $\boldsymbol{Px} \succ \boldsymbol{x}$, but $\boldsymbol{Px} \sim \boldsymbol{x}$ by $\mathcal{P}$-Anonymity.

Sets $\mathcal{P}$, $\mathcal{S}$, and $\mathcal{F}$ of permutations are groups of permutations, but $\mathcal{C}$ is not. Since $\mathcal{F} \subset \mathcal{S} \subset \mathcal{C}$, $\mathcal{S}$ and $\mathcal{F}$ are groups of cyclic permutations.

By the standard application of Zorn's lemma (as used in proving the existence of a basis for a vector space), it can be checked that there exists a maximal group of cyclic permutations. To demonstrate this, let $\mathfrak{Q}$ be the set of all groups $\mathcal{Q}$ of cyclic permutations. Note that the set inclusion $\subseteq$ is a partial order on $\mathfrak{Q}$.[17] We say that (i) a set A of objects is *partially ordered* by a binary relation $\sqsubseteq$ if $\sqsubseteq$ is defined on A and $\sqsubseteq$ is a partial order, and (ii) a set A of objects is *totally ordered* by a binary relation $\sqsubseteq$ if A is partially ordered by $\sqsubseteq$ and $\sqsubseteq$ is complete. Given a set A which is partially ordered by a binary relation $\sqsubseteq$ and given a subset $B \subseteq A$, we say that (i) B has an *upper bound* in A if there exists $a \in A$ such that $b \sqsubseteq a$ for all $b \in B$, and (ii) $a \in A$ is a *maximal element* if there is no $b \in A$ such that $a \sqsubset b$, where $\sqsubset$ is the asymmetric part of $\sqsubseteq$.

Zorn's lemma is stated as follows (see, for example, Roman (2008, p. 10)).

Zorn's lemma. *Let A be a set of objects and suppose that A is partially ordered by a binary relation $\sqsubseteq$. If every non-empty totally ordered subset of A has an upper bound in A, then A has a maximal element.*

The existence of a maximal group of cyclic permutations is stated as the following proposition.

Proposition 3. *$\mathfrak{Q}$ has a maximal element with respect to $\subseteq$.*

We now demonstrate the proof of this proposition by using Zorn's lemma.

Proof of Proposition 3. Note that $\mathfrak{Q}$ is partially ordered by $\subseteq$. We show that the antecedent of Zorn's lemma is satisfied. Let $\hat{\mathfrak{Q}}$ be any non-empty totally ordered subset of $\mathfrak{Q}$. We define $\mathcal{Q}^*$ by

$$\mathcal{Q}^* = \bigcup_{\mathcal{Q} \in \hat{\mathfrak{Q}}} \mathcal{Q}. \tag{3}$$

We show that

$$\begin{cases} \mathcal{Q}^* \in \mathfrak{Q}, \\ \mathcal{Q} \subseteq \mathcal{Q}^* \text{ for all } \mathcal{Q} \in \hat{\mathfrak{Q}}, \end{cases}$$

i.e., $\hat{\mathfrak{Q}}$ has an upper bound in $\mathfrak{Q}$.

It is straightforward from (3) that $\mathcal{Q} \subseteq \mathcal{Q}^*$ for all $\mathcal{Q} \in \hat{\mathfrak{Q}}$. We prove that $\mathcal{Q}^* \in \mathfrak{Q}$. First, to show that any $\boldsymbol{P} \in \mathcal{Q}^*$ is cyclic, choose any $\boldsymbol{P} \in \mathcal{Q}^*$. By (3), there exists $\mathcal{Q} \in \hat{\mathfrak{Q}}$ such that $\boldsymbol{P} \in \mathcal{Q}$. Since $\hat{\mathfrak{Q}} \subseteq \mathfrak{Q}$, $\boldsymbol{P}$ is cyclic.

Next, we show that $\mathcal{Q}^*$ satisfies Properties 1, 2, and 3, i.e., $\mathcal{Q}^*$ is a group of permutations. To verify Property 1, choose any $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q}^*$. By (3), there exist $\mathcal{Q}^1, \mathcal{Q}^2 \in \hat{\mathfrak{Q}}$ such that $\boldsymbol{P} \in \mathcal{Q}^1$ and $\boldsymbol{Q} \in \mathcal{Q}^2$. Since $\hat{\mathfrak{Q}}$ is totally ordered by $\subseteq$, either $\mathcal{Q}^1 \subseteq \mathcal{Q}^2$ or $\mathcal{Q}^2 \subseteq \mathcal{Q}^1$ holds. Without loss of generality, we assume $\mathcal{Q}^1 \subseteq \mathcal{Q}^2$. Then, $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{Q}^2$. Since $\hat{\mathfrak{Q}} \subseteq \mathfrak{Q}$, $\mathcal{Q}^2$ is a group of permutations. Thus, $\boldsymbol{PQ} \in \mathcal{Q}^2$. Since $\mathcal{Q}^2 \subseteq \mathcal{Q}^*$, $\boldsymbol{PQ} \in \mathcal{Q}^*$.

Next, to prove Property 2, choose any $\boldsymbol{P} \in \mathcal{Q}^*$. By (3), there exists $\mathcal{Q} \in \hat{\mathfrak{Q}}$ such that $\boldsymbol{P} \in \mathcal{Q}$. Since $\hat{\mathfrak{Q}} \subseteq \mathfrak{Q}$, $\mathcal{Q}$ is a group of permutations. Thus, $\boldsymbol{I} \in \mathcal{Q}$ and $\boldsymbol{PI} = \boldsymbol{IP} = \boldsymbol{P}$. Since $\mathcal{Q} \subseteq \mathcal{Q}^*$, $\boldsymbol{I} \in \mathcal{Q}^*$.

Finally, we show that $\mathcal{Q}^*$ satisfies Property 3. Choose any $\boldsymbol{P} \in \mathcal{Q}^*$. By (3), there exists $\mathcal{Q} \in \hat{\mathfrak{Q}}$ such that $\boldsymbol{P} \in \mathcal{Q}$. Since $\hat{\mathfrak{Q}} \subseteq \mathfrak{Q}$, $\mathcal{Q}$ is a group of permutations. Thus, there exists $\boldsymbol{P}' \in \mathcal{Q}$ such that $\boldsymbol{P}'\boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}' = \boldsymbol{I}$. Since $\mathcal{Q} \subseteq \mathcal{Q}^*$, $\boldsymbol{P}' \in \mathcal{Q}^*$.

Since $\mathcal{Q}^*$ is an upper bound for $\hat{\mathfrak{Q}}$ and $\hat{\mathfrak{Q}}$ is arbitrarily chosen, it follows from Zorn's lemma that $\mathfrak{Q}$ has a maximal element with respect to $\subseteq$. □

By Proposition 3, there exists a maximal group of cyclic permutations. However, our proof relies on the use of Zorn's lemma and a maximal group of cyclic permutation is not explicitly constructed. Lauwers (2012) proves that the existence of a maximal group of cyclic permutations must involve the use of non-constructive mathematics (the existence of a free ultrafilter) and any maximal group of cyclic permutations cannot be explicitly described. This implies that $\mathcal{S}$ is not a maximal group of cyclic permutations. However, in view of non-constructibility of a maximal group of cyclic permutations, **SA** is important ranther than an insufficient formalization of equal treatment of generations.

Notes

1) An SWQ is a reflexive and transitive binary relation. The definitions of these notions are given in Sect. II 1..
2) See also the related work of Basu and Mitra (2007) on the utilitarian catching-up SWQ.
3) More precisely, Asheim and Tungodden (2004) establish axiomatic characterizations of the utilitarian and leximin catching-up SWQs using **SP**, **FA** and some additional axioms regarding equity and consitency priperties of the evaluation.
4) On this, see the review article by Asheim (2010).
5) The approach of evaluating utility streams is termed *welfarism*. In this approach, temporal social states or consumption levels that underly utility streams are excluded from relevant information for the evaluation of intergenerational distributional problems. An axiomatic justification for welfarism is presented by d'Aspremont (2007).
6) Svensson (1980) proves the existence of an SWO that (i) includes the utilitarian catching-up SWQ as a subrelation and (ii) satisfies Strong Pareto and Finite Anonymity. However, his proof uses Arrow's (1963) variant of Szpilrajn's (1930) lemma, and the SWO is not explicitly constructed. Lauwers (2010) and Zame (2007) prove that any SWO that satisfies these axioms must involve the use of non-constructive mathematics such like Axiom of Choice and it cannot be explicitly described. The conjecture of this impossibility is presented by Fleurbaey and Michel (2003).
7) In Asheim and Tungodden (2004), the leximin catching-up SWQ is called *S-leximin* SWQ.
8) This axioms is a weakening of Strong Preference Continuity considered in Asheim and Tungodden (2004). Further, it is weaker than Strong Consistency presented in Basu and Mitra (2007).
9) See also Basu and Mitra (2007).
10) Hereafter, we will use the terms "permutation" and "permutation matrix" interchangeably. For a detail explanation of the relationship between a permutation and a permutation matrix, see, e.g., Mitra and Basu (2007).
11) For any $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{P}$, the product $\boldsymbol{PQ}$ is defined by $(r_{ij})_{i,j \in \mathbb{N}}$ with $r_{ij} = \Sigma_{k \in \mathbb{N}} p_{ik} q_{kj}$.
12) This impossibility is also suggested by Svensson (1980, p. 1252).

13) On this, see, e.g., Asheim and Banerjee (2010), Banerjee (2006), Fleurbaey and Michel (2003), Lauwers (1997b; 2010), Mitra and Basu (2007), Kamaga, Kamijo, Shinotsuka (2009), and Kamaga and Kojima (2009; 2010). For SWQs that satisfy **FA** but violate **SA**, see, e.g., Asheim, d'Aspremont, and Banerjee (2010), Basu and Mitra (2007), and Bossert, Sprumont, and Suzumura (2007).

14) Strong Monotonicity for Infinite Generations is called Infinite Pareto Priciple in Crespo, Nuñez, and Rincón-Zapatero (2009).

15) In algebra, a set of objects is said to be a group if it satisfies *associativity* in addition to Properties 1, 2, and 3. Given a set $\mathcal{O}$ of objects, $\mathcal{O}$ together with an operation $\circ$ satisfy associativity if, for all $A, B, C \in \mathcal{O}, A \circ (B \circ C) = (A \circ B) \circ C$. Since any set of permutations is associative, we omit associativity in the definition of a group of permutations.

16) Such a permutation is called *variable-step permutation*.

17) A binary relation $\sqsubseteq$ on a set of objects A is anti-symmetric if, for all $a, b \in A$, $a = b$ whenever $a \sqsubseteq b$ and $b \sqsubseteq a$. A binary relation $\sqsubseteq$ on a set of objects A is a partial order if it is an anti-symmetric quasi-ordering.

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